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ACADEMIC PRESS

Journal of Approximation Theory 122 (2003) 224–240

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Strong asymptotics in Lagrange interpolation with equidistant nodes

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Received 11 April 2002; accepted in revised form 3 April 2003

Communicated by József Szabados

Abstract

In this paper we prove three conjectures of Revers on Lagrange interpolation for $f_\lambda(t) = |t|^\lambda, \lambda > 0$, at equidistant nodes. In particular, we describe the rate of divergence of the Lagrange interpolants $L_N(f_\lambda, t)$ for $0 < |t| < 1$, and discuss their convergence at $t = 0$. We also establish an asymptotic relation for $\max_{|t| \leqslant 1} | |t|^\lambda - L_N(f_\lambda, t)|$. The proofs are based on strong asymptotics for $|t|^\lambda - L_N(f_\lambda, t),\ 0 \leqslant |t| < 1$.

Keywords: Lagrange interpolation; Equidistant nodes; Strong asymptotics

1. Introduction

Let $\mathscr{P}_N$ be the set of all algebraic polynomials of degree at most N, and let $L_N(f, \cdot) \in \mathscr{P}_N$ be the Lagrange interpolation polynomial to a continuous function f on $[-1, 1]$ associated with the equidistant nodes

$$t_{j,N} := -1 + 2j/N, \quad j = 0, 1, \ldots, N, \quad N = 1, 2, \ldots \ . \tag{1.1}$$

The limit behavior of $L_N(f_\lambda, t)$, where $f_\lambda(t) := |t|^\lambda, \lambda > 0, t \in (-1, 1)$, and other related problems have attracted much attention of several generations of mathematicians (see [1–5,8,11–18]). The story begins, like many others in approximation theory, with Bernstein in 1916. Searching for an "elementary

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0021-9045/03/$ - see front matter © 2003 Published by Elsevier Science (USA).
doi:10.1016/S0021-9045(03)00070-4

example" of a function whose Lagrange interpolation polynomials diverge everywhere, he outlined in [1] (see also the reprinted version [2]) the proof of the following statement: the sequence of Lagrange interpolation polynomials to $|t|$ at nodes (1.1) diverges "at any interval" of $[-1, 1]$. In fact, he proved only the estimate

$$|L_{2n}(f_1, t)| \geqslant e^{nt^2}/(8n^3), \tag{1.2}$$

where t is a midpoint between two consecutive nodes. The detailed proof of the relation

$$\limsup_{N\to\infty} |L_N(f_\lambda, t)| = \infty, \quad 0<|t|<1, \tag{1.3}$$

for $\lambda = 1$ can be found in [13, pp. 30–35].

In his paper, Bernstein did not discuss the behavior of $L_N(f_1, 0)$ as $N\to\infty$, probably because 0 is a node for all even $N>0$. The asymptotic formula

$$\lim_{N\to\infty} L_N(f_1, 0) = 0 \tag{1.4}$$

was established in 1939 by Berman in his student term paper (see [13, pp. 34, 35]).

Much work has been done in the 1990s and 2000 to extend relations (1.3) and (1.4) to $\lambda \neq 1$ and to find the asymptotic behavior of $L_N(f_\lambda, t) - |t|^\lambda$ for $t\in(-1, 1)$. In particular, Revers [16] showed that (1.3) holds true for $\lambda\in(0, 1)$ and established in [17] the surprising formula

$$\lim_{N=2n-1\to\infty} N^\lambda L_N(f_\lambda, 0) = 2(2/\pi)^{\lambda+1}\sin(\pi\lambda/2)\int_0^\infty \frac{y^{\lambda-1}}{e^y+e^{-y}}\,dy, \tag{1.5}$$

where $\lambda\in(0, 1]$.

Inequality (1.2) shows that the rate of divergence of the sequence $\{L_N(f_1, t)\}_{N=1}^\infty$ depends on the location of t in $[-1, 1]$. Byrne et al. [5] amplified (1.2) by proving the following nth root asymptotic relation for $0<|t|<1$ and $\lambda = 1$:

$$\limsup_{N\to\infty} N^{-1}\log|\,|t|^\lambda - L_N(f_\lambda, t)| = (1/2)((1+t)\log(1+t) + (1-t)\log(1-t)). \tag{1.6}$$

The extension of (1.6) to $\lambda = 3$ was given in [15].

Li and Mohapatra [11] showed that (1.6) holds true for $\lambda = 1$ and almost every $t\in[-1, 1]$ with $\limsup_{N\to\infty}$ replaced by $\lim_{N=p_k+1\to\infty}$, where $\{p_k\}_{k=1}^\infty$ is the increasing sequence of all positive prime numbers.

Recently Revers, motivated by numerical calculations [16,17] and by aesthetic reasons [15], conjectured that relations (1.3), (1.5), and (1.6) remain valid for all relevant $\lambda>0$.

In this paper we prove these conjectures (Theorem 1, Corollaries 1 and 2). Moreover, we establish strong asymptotics for $|t|^\lambda - L_N(f_\lambda, t)$, where $t\in(-1, 1)$ (Theorems 2, 4 and 5). As corollaries, we strengthen and generalize the result of Li and Mohapatra (Theorem 3) and obtain an asymptotic relation for $\max_{|t|\leqslant 1}|\,|t|^\lambda - L_N(f_\lambda, t)|$ (Theorem 6).

Notation. Throughout the paper λ is a real number, $\lambda \neq 0, 2, \dots$, and C denotes a positive constant independent of $M, N, n, t, y, \varepsilon$. The same symbol does not necessarily denote the same constant in different occurrences. We also make use of the following functions for $t \in [-1, 1]$ and constants for $\lambda > 0$:

$$\varphi_N(t) := \sqrt{1-t^2}((1+t)^{1+t}(1-t)^{1-t})^{N/2},$$

$$s(t) := \begin{cases} \cos\frac{\pi}{2m}, & t = p/m, (p,m) = 1, m \text{ is odd}, |p| \in \mathbb{N}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c(t) := \begin{cases} \cos\frac{\pi}{2m}, & t = p/m, (p,m) = 1, p \text{ is odd}, |m| \in \mathbb{N}, \\ 1, & \text{otherwise}, \end{cases}$$

$$C_1(\lambda) := \int_0^\infty \frac{y^{\lambda-1}}{e^y + e^{-y}}\, dy = \Gamma(\lambda) \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-\lambda},$$

$$C_2(\lambda) := \int_0^\infty \frac{y^{\lambda}}{e^y - e^{-y}}\, dy = \Gamma(\lambda+1) \sum_{k=0}^{\infty} (2k+1)^{-(\lambda+1)}.$$

2. Statement of main results

We first discuss the asymptotic behavior of $L_N(f_\lambda, 0), N \in \mathbb{N}$. Since $L_{2n}(f_\lambda, 0) = 0$ for $\lambda > 0$ and all $n \in \mathbb{N}$, here we study the asymptotic behavior of $L_{2n-1}(f_\lambda, 0)$, $n \in \mathbb{N}$.

Theorem 1. *If* $\lambda > 0$, *then*

$$\lim_{N=2n-1 \to \infty} N^\lambda L_N(f_\lambda, 0) = 2(2/\pi)^{\lambda+1} \sin(\pi\lambda/2) C_1(\lambda). \tag{2.1}$$

Next we establish the rate of divergence of the sequence $| |t|^\lambda - L_N(f_\lambda, t)|$ for $0 < |t| < 1$.

Theorem 2. *Let* $t \in (-1, 0) \cup (0, 1)$ *be a fixed point.*

(a) *If* $\lambda > -2$, *then*

$$\limsup_{N=2n-1 \to \infty} ((\pi N/2)^{\lambda+2}/\varphi_N(t))\, | |t|^\lambda - L_N(f_\lambda, t)|$$
$$= (4/\pi)|\sin(\pi\lambda/2)| C_1(\lambda+2) t^{-2} c(t). \tag{2.2}$$

(b) *If* $\lambda > 0$, *then*

$$\limsup_{N=2n \to \infty} ((\pi N/2)^{\lambda+1}/\varphi_N(t))\, | |t|^\lambda - L_N(f_\lambda, t))|$$
$$= (4/\pi)|\sin(\pi\lambda/2)| C_2(\lambda) |t|^{-1} s(t). \tag{2.3}$$

As immediate consequences of Theorem 2, we extend (1.6) and(1.3) to $\lambda > 0$.

Corollary 1. *If* $0<|t|<1$ *and* $\lambda>0$, *then* (1.6) *holds.*

Corollary 2. *If* $\lambda>0$, *then* (1.3) *is valid.*

Next we show that (1.6) holds for almost all $t\in[-1,1]$ with $\limsup_{N\to\infty}$ replaced by $\lim_{N\to\infty}$.

Theorem 3. *For* $\lambda>0$ *and almost all* $t\in[-1,1]$,

$$\lim_{N\to\infty} N^{-1}\log|\,|t|^\lambda - L_N(f_\lambda,t)| = (1/2)((1+t)\log(1+t)+(1-t)\log(1-t)).$$

The following strong asymptotics play a crucial role in the proofs of Theorems 1, 2, and 3 and are interesting in themselves.

Theorem 4. (a) *If* $0\leqslant|t|<1$, $\lambda>0$, *and* $N=2n-1, n\in\mathbb{N}$, *then*

$$|t|^\lambda - L_N(f_\lambda,t) = -(4/\pi)\sin(\pi\lambda/2)(\pi N/2)^{-\lambda}\cos(\pi Nt/2)\varphi_N(t) \times\int_0^\infty \frac{y^{\lambda+1}}{((\pi Nt/2)^2+y^2)(e^y+e^{-y})}\,dy(1+\alpha_{N,1}(t)), \tag{2.4}$$

where $|\alpha_{N,1}(t)|\leqslant C(N^{-1/3}+(N(1-t^2))^{-1})$.

(b) *If* $0\leqslant|t|<1$, $\lambda>0$, *and* $N=2n$, $n\in\mathbb{N}$, *then*

$$|t|^\lambda - L_N(f_\lambda,t) = (4/\pi)\sin(\pi\lambda/2)(\pi N/2)^{-\lambda+1}t\sin(\pi Nt/2)\varphi_N(t) \times\int_0^\infty \frac{y^{\lambda}}{((\pi Nt/2)^2+y^2)(e^y-e^{-y})}\,dy(1+\alpha_{N,2}(t)), \tag{2.5}$$

where $|\alpha_{N,2}(t)|\leqslant C(N^{-1/3}+(N(1-t^2))^{-1})$.

(c) *If* $0<|t|<1$, $\lambda>-2$, *and* $N=2n-1$, $N\in\mathbb{N}$, *then*

$$|t|^\lambda - L_N(f_\lambda,t) = -(4/\pi)\sin(\pi\lambda/2)C_1(\lambda+2)(\pi N/2)^{-(\lambda+2)}t^{-2} \times\cos(\pi Nt/2)\varphi_N(t)(1+\alpha_{N,3}(t)), \tag{2.6}$$

where $|\alpha_{N,3}(t)|\leqslant C(N^{-1/3}(1+(Nt^2)^{-1})+(N(1-t^2))^{-1})$.

(d) *If* $0<|t|<1$, $\lambda>0$, *and* $N=2n$, $N\in\mathbb{N}$, *then*

$$|t|^\lambda - L_N(f_\lambda,t) = (4/\pi)\sin(\pi\lambda/2)C_2(\lambda)(\pi N/2)^{-(\lambda+1)}t^{-1} \times\sin(\pi Nt/2)\varphi_N(t)(1+\alpha_{N,4}(t)), \tag{2.7}$$

where $|\alpha_{N,4}(t)|\leqslant C(N^{-1/3}(1+(Nt^2)^{-1})+(N(1-t^2))^{-1})$.

To prove Theorem 4, we apply Bernstein's approach, developed in 1937 for interpolation with the Chebyshev nodes [3] (see also [8,20]), to equidistant interpolation.

Theorem 4 provides the uniform asymptotics for $|t|^\lambda - L_N(f_\lambda, t)$ in the interval $|t| \leqslant 1 - \alpha_N/N$, where $0 \leqslant \alpha_N \leqslant N$ and $\lim_{N\to\infty} \alpha_N = \infty$. The asymptotic formulae for $|t|^\lambda - L_N(f_\lambda, t)$ at $t = \pm(1 - \alpha_N/N)$, where $\lim_{N\to\infty} \alpha_N = 0$, are given below.

Theorem 5. *Let* $|t| = 1 - \alpha_N/N$, *where* $\lim_{N\to\infty} \alpha_N = 0$.

(a) *If* $N = 2n - 1, n \in \mathbb{N}$, *and* $\lambda > -2$, *then for* $N \to \infty$,

$$|t|^\lambda - L_N(f_\lambda, t) = (-1)^{(N+1)/2} \sin(\pi\lambda/2) C_1(\lambda + 2)(\pi N/2)^{-(\lambda+5/2)} \times \alpha_N N^{-\alpha_N/2} 2^{N+1}(1 + o(1)). \tag{2.8}$$

(b) *If* $N = 2n, n \in \mathbb{N}$, *and* $\lambda > 0$, *then for* $N \to \infty$,

$$|t|^\lambda - L_N(f_\lambda, t) = (-1)^{N/2+1} \sin(\pi\lambda/2) C_2(\lambda)(\pi N/2)^{-(\lambda+3/2)} \times \alpha_N N^{-\alpha_N/2} 2^{N+1}(1 + o(1)). \tag{2.9}$$

Finally, we use Theorems 4 and 5 to establish an asymptotic relation for the approximation error.

Theorem 6. *If* $\lambda > 0$, *then*

$$\Delta_{N,\lambda} := \max_{|t|\leqslant 1} \left| |t|^\lambda - L_N(f_\lambda, t) \right| = \begin{cases} A_1 N^{-(\lambda+5/2)} 2^N / \log N (1 + o(1)), & N = 2n - 1 \to \infty, \\ A_2 N^{-(\lambda+3/2)} 2^N / \log N (1 + o(1)), & N = 2n \to \infty, \end{cases} \tag{2.10}$$

where

$$A_1 = (4/e)|\sin(\pi\lambda/2)| C_1(\lambda + 2)(\pi/2)^{-(\lambda+5/2)}, \tag{2.11}$$

$$A_2 = (4/e)|\sin(\pi\lambda/2)| C_2(\lambda)(\pi/2)^{-(\lambda+3/2)}. \tag{2.12}$$

Remark 1. Theorem 2 implies that for $\lambda > 0$ and $0 < |t| < 1$,

$$\limsup_{N=2n\to\infty} |L_N(f_\lambda, t)| = \limsup_{N=2n\to\infty} |L_N(f_\lambda, t)| = \infty.$$

This solves the problem on the behavior of $L_{2n-1}(f_\lambda, t)$ as $n \to \infty$ for $\lambda \in (0, 1]$, posed in [16].

Remark 2. We note that the constant on the right-hand side of (2.1) is surprisingly related to the constant in Lagrange interpolation to f_λ with the Chebyshev nodes (see [3,8,14]).

$$\lim_{N\to\infty} N^\lambda \max_{|t|\leqslant 1} \left| |t|^\lambda - L_N^*(f_\lambda, t) \right| = (4/\pi)|\sin(\pi\lambda/2)| C_1(\lambda).$$

Revers [17] believes that the constant in (2.1) is related to the Bernstein constant $B_\lambda := \lim_{N\to\infty} N^\lambda \inf_{P\in\mathscr{P}_N} \max_{|t|\leqslant 1} |f_\lambda(t) - P(t)|$.

Remark 3. Theorem 6 shows that the growth of $\Delta_{N,\lambda}$ is $N^{\lambda+1/2}$ slower than the order of the magnitude of the Lebesgue constant $||L_N||$ whose asymptotic behavior

$$||L_N|| \sim 2^{N+1}/(eN(\log N+\gamma)) \quad \text{as } N\to\infty,$$

was established by Schönhage [19]. Here $\gamma = 0.577\ldots$ denotes Euler's constant.

Remark 4. The exponential factor in Theorems 2 and 4 can be expressed through the potential corresponding to the uniform distribution on $[-1,1]$ (see [12]):

$$((1+t)^{1+t}(1-t)^{1-t})^{N/2} = \exp\left((N/2)\int_{-1}^{1}\log|t-y|\,dy + N\right).$$

Remark 5. Theorems 2–6 are new even for $\lambda = 1$.

3. Proof of Theorem 4

The proof follows Lemma 1 in [8] (see also [3, pp. 92, 98–100]), though the equidistant nodes require more detailed analysis than the Chebyshev ones.

To prove the theorem, we need two lemmas. The proof of the first one is outlined in [3, p. 92], and a special case of the lemma is given in [8]. Here, for the convenience of the reader, we give a proof of the following result.

Lemma 1. *Let $P_{m-1}\in\mathscr{P}_{m-1}$ be the Lagrange interpolation polynomial to $(1-x)^s$ on $[-1,1]$ at the nodes $\{x_k\}_{k=1}^m$, $-1\leqslant x_1<,\cdots<x_m\leqslant 1$, and let $Q_d(x) := \prod_{k=1}^d (x-x_k)$.*

(a) *If $x_m<1$ and $m>s>-1$, then for $x\in[-1,1]$,*

$$(1-x)^s - P_{m-1}(x) = -(1/\pi)\sin\pi s\, Q_m(x)\int_1^\infty \frac{(z-1)^s}{(z-x)Q_m(z)}\,dz. \tag{3.1}$$

(b) *If $x_m = 1$ and $m>s>0$, then for $x\in[-1,1]$,*

$$(1-x)^s - P_{m-1}(x) = (1/\pi)\sin\pi s\,(1-x)Q_{m-1}(x)\int_1^\infty \frac{(z-1)^{s-1}}{(z-x)Q_{m-1}(z)}\,dz. \tag{3.2}$$

Proof. We first prove statement (a) of the lemma. Let $P_{m-1,a}\in\mathscr{P}_{m-1}$ be the interpolation polynomial to $(a-x)^s$ on $[-1,1]$ at $\{x_k\}_{k=1}^m$, where $a>1$. By the Hermite error formula for Lagrange interpolation,

$$(a-x)^s - P_{m-1,a}(x) = \frac{Q_m(x)}{2\pi i}\lim_{M\to\infty}\lim_{\varepsilon\to 0}\int_{D_{M,\varepsilon}}\frac{(a-z)^s}{(z-x)Q_m(z)}\,dz, \tag{3.3}$$

where $(a-z)^s$ takes positive values for real $z<a, s>-1$. Here, $D_{M,\varepsilon} = C_{M,\varepsilon}\cup C_\varepsilon\cup D_\varepsilon\cup D_{-\varepsilon}$ is a contour in $\mathbf{C}$, oriented in a positive sense, where M and

$\varepsilon, M > a > (a-1)/2 > \varepsilon > 0$, are fixed numbers and

$$C_{M,\varepsilon} := \{z : |z| = M, \arcsin(\varepsilon/M) \leqslant |\arg z| \leqslant \pi\},$$

$$C_\varepsilon := \{z : |z-a| = \varepsilon,\ \pi/2 \leqslant |\arg z| \leqslant \pi\},$$

$$D_{\pm\varepsilon} := \{z = x \pm i\varepsilon : a \leqslant x \leqslant \sqrt{M^2 - \varepsilon^2}\}.$$

Since the function $h(z) := \frac{(a-z)^s}{(z-x)Q_m(z)}$ satisfies the conditions

$$\max_{z \in C_{M,\varepsilon}} |h(z)| \leqslant CM^{s-m-1}, \quad \max_{z \in C_\varepsilon} |h(z)| \leqslant C\varepsilon^s,$$

we have

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} \int_{C_{M,\varepsilon}} h(z)\,dz = \lim_{M\to\infty} \lim_{\varepsilon\to 0} \int_{C_\varepsilon} h(z)\,dz = 0. \tag{3.4}$$

Next, by the limit relation

$$\lim_{\varepsilon\to 0} (a-(x+i\varepsilon))^s - (a-(x-i\varepsilon))^s = -2i\sin\pi s(x-a)^s, \quad x \geqslant a,$$

we obtain

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0}\left(\int_{D_\varepsilon} h(z)\,dz + \int_{D_{-\varepsilon}} h(z)\,dz\right) = -2i\sin\pi s \int_a^\infty h(z)\,dz. \tag{3.5}$$

Then (3.3)–(3.5) yield the integral representation

$$(a-x)^s - P_{m-1,a}(x) = (1/\pi)\sin\pi s\, Q_m(x) \int_a^\infty \frac{(z-a)^s}{(z-x)Q_m(z)}\,dz \tag{3.6}$$

for $x_m < 1$. Finally, making the substitution $z = au$ in this integral and letting $a \to 1+$ in (3.6), we obtain (3.1), by the Lebesgue-dominated convergence theorem.

Statement (b) can be proved similarly. □

In the next lemma we study the asymptotic behavior of some polynomials.

Lemma 2. (a) *If* $n \in \mathbb{N}$ *and* $|y| \leqslant n^{1/3}$, *then*

$$\left(\prod_{k=1}^n \left(1 + \frac{4y^2}{\pi^2(2k-1)^2}\right)\right)^{-1} = (\cosh y)^{-1}(1 + \beta_{n,1}(y)), \tag{3.7}$$

$$\left(\prod_{k=1}^n \left(1 + \frac{y^2}{\pi^2 k^2}\right)\right)^{-1} = y(\sinh y)^{-1}(1 + \beta_{n,2}(y)), \tag{3.8}$$

where $0 \leqslant \beta_{n,j}(y) \leqslant Cn^{-1/3}$, $j = 1, 2$.

(b) *If* $n \in \mathbb{N}$ *and* $|t| < 1$, *then*

$$\prod_{k=1}^n \left(1 - \left(\frac{2n-1}{2k-1}t\right)^2\right) = \cos(\pi(2n-1)t/2)\varphi_{2n-1}(t)(1 + \beta_{n,3}(t)), \tag{3.9}$$

$$\prod_{k=1}^{n}\left(1-\left(\frac{n}{k}t\right)^2\right)=\frac{\sin(\pi nt)}{\pi nt}\varphi_{2n}(t)(1+\beta_{n,4}(t)), \tag{3.10}$$

where $|\beta_{n,j}(t)| \leqslant C(n(1-t^2))^{-1}$, $j=3,4$.

Proof. (a) Using the product formulae for $\cosh y$ and $\sinh y$ [9, Section 1.431], we obtain

$$\begin{aligned}1+\beta_{n,1}(y) &:= \cosh y\left(\prod_{k=1}^{n}\left(1+\frac{4y^2}{\pi^2(2k-1)^2}\right)\right)^{-1}\\ &= \prod_{k=n+1}^{\infty}\left(1+\frac{4y^2}{\pi^2(2k-1)^2}\right)\\ &\leqslant \exp\left((4y^2/\pi^2)\sum_{k=n+1}^{\infty}(2k-1)^{-2}\right)\leqslant \exp(Cy^2/n)\leqslant 1+Cn^{-1/3},\end{aligned}$$

$$\begin{aligned}1+\beta_{n,2}(y) &:= \sinh y\left(y\prod_{k=1}^{n}\left(1+\frac{y^2}{\pi^2k^2}\right)\right)^{-1}\\ &= \prod_{k=n+1}^{\infty}\left(1+\frac{y^2}{\pi^2k^2}\right)\leqslant 1+Cn^{-1/3}.\end{aligned}$$

These inequalities yield (3.7) and (3.8).

(b) We first note that the asymptotic $(1+y/n)^n = e^y(1+O(1/n))$ holds uniformly in every interval $[-C, C]$, where C is a fixed constant. Then using [7, Section 1.2] and taking account of the asymptotic formula for the gamma function [7, Section 1.18], we obtain after easy manipulations

$$\begin{aligned}&\left(\cos\frac{\pi(2n-1)t}{2}\right)^{-1}\prod_{k=1}^{n}\left(1-\left(\frac{2n-1}{2k-1}t\right)^2\right)\\ &=\frac{\Gamma(n(1+t)+\frac{1-t}{2})\Gamma(n(1-t)+\frac{1+t}{2})}{(\Gamma(n+1/2))^2}\\ &=\frac{(n(1+t)+\frac{1-t}{2})^{n(1+t)+(1-t)/2}(n(1-t)+\frac{1+t}{2})^{n(1-t)+(1+t)/2}}{n(n+1/2)^{2n}\sqrt{1-t^2}}\left(1+O\left(\frac{1}{n(1-t^2)}\right)\right)\\ &=\frac{en^{2n}(1+t)^{n(1+t)+(1-t)/2}(1-t)^{n(1-t)+(1+t)/2}}{(n+1/2)^{2n}\sqrt{1-t^2}}\left(1+O\left(\frac{1}{n(1-t^2)}\right)\right)\\ &=\varphi_{2n-1}(t)\left(1+O\left(\frac{1}{n(1-t^2)}\right)\right).\end{aligned}$$

Thus (3.9) follows. Similarly by [7, Sections 1.2 and 1.18],

$$\prod_{k=1}^{n}\left(1-\left(\frac{n}{k}t\right)^2\right)$$
$$=\frac{\sin(\pi nt)}{\pi nt}(n!)^{-2}\Gamma(n(1+t)+1)\Gamma(n(1-t)+1)$$
$$=\frac{\sin(\pi nt)}{\pi nt}\frac{(n(1+t)+1)^{n(1+t)+1}(n(1-t)+1)^{n(1-t)+1}}{n\sqrt{1-t^2}}\left(1+O\left(\frac{1}{n(1-t^2)}\right)\right)$$
$$=\frac{\sin(\pi nt)}{\pi nt}\varphi_{2n}(t)\left(1+O\left(\frac{1}{n(1-t^2)}\right)\right).$$

This yields (3.10). □

Proof of Theorem 4. (a) and (c): Let $t\in[-1,1], \lambda>-2$, and $N=2n-1, N\in\mathbb{N}$. We first consider the following nodes:

$$x_{n-k+1}:=1-2((2k-1)/(2n-1))^2, \quad k=1,\dots,n.$$

Then $x_n<1$ and by Lemma 1(a) for $m=n>s>-1$,

$$(1-x)^s-P_{n-1}(x)$$
$$=-(1/\pi)\sin\pi s\prod_{k=1}^{n}\left(x-1+2\left(\frac{2k-1}{2n-1}\right)^2\right)$$
$$\int_1^\infty\frac{(z-1)^s}{(z-x)\prod_{k=1}^n(z-1+2(\frac{2k-1}{2n-1})^2)}dz.$$

Making the substitution $z=1+2(\frac{2y}{\pi(2n-1)})^2$ in this integral, we arrive at the identity

$$(1-x)^s-P_{n-1}(x)$$
$$=-(2^{s+2}/\pi)\sin\pi s\,(\pi(2n-1)/2)^{-(2s+2)}\frac{(2n-1)^{2n}}{2^n\prod_{k=1}^n(2k-1)^2}$$
$$\times\prod_{k=1}^{n}\left(x-1+2\left(\frac{2k-1}{2n-1}\right)^2\right)$$
$$\times\int_0^\infty\frac{y^{2s+1}}{(1-x+2(\frac{2y}{\pi(2n-1)})^2)\prod_{k=1}^n(1+\frac{4y^2}{\pi^2(2k-1)^2})}dy. \tag{3.11}$$

Next we make the substitutions $x=1-2t^2$ and $s=\lambda/2$ in (3.11) and note that

$$L_{2n-1}(f_\lambda,t)=2^{-s}P_{n-1}(1-2t^2), \quad t\in[-1,1]. \tag{3.12}$$

Thus (3.11) and (3.12) yield

$$|t|^{\lambda} - L_{2n-1}(f_{\lambda}, t)$$
$$= -(2/\pi)\sin(\pi\lambda/2)(\pi(2n-1)/2)^{-(\lambda+2)}$$
$$\times \prod_{k=1}^{n}\left(1-\left(\frac{2n-1}{2k-1}t\right)^{2}\right)\int_{0}^{\infty}\frac{y^{\lambda+1}}{(t^{2}+(\frac{2y}{\pi(2n-1)})^{2})\prod_{k=1}^{n}(1+\frac{4y^{2}}{\pi^{2}(2k-1)^{2}})}\,dy\ . \tag{3.13}$$

To prove statements (a) and (c), we need to find the asymptotic behavior of the integral $I_n(t)$ on the right-hand side of (3.13). Splitting $I_n(t)$ into two integrals, we obtain

$$I_n(t) = \int_0^{n^{1/3}} + \int_{n^{1/3}}^{\infty} = I_{n,1}(t) + I_{n,2}(t). \tag{3.14}$$

Note that for $0 \leqslant |t| < 1$, $\lambda > -2$, and $n \geqslant m := [(\lambda+7)/2] + 1$, where $[x]$ denotes the integer part of x, we have

$$I_{n,2}(t) \leqslant Cn^{2}\int_{n^{1/3}}^{\infty}\frac{y^{\lambda-1}}{\prod_{k=1}^{m}(1+\frac{4y^{2}}{\pi^{2}(2k-1)^{2}})}\,dy \leqslant Cn^{2+(\lambda-2m)/3} \leqslant Cn^{-1/3}. \tag{3.15}$$

Hence for $t \in [0, 1)$ and $\lambda > 0$,

$$I_{n,2}(t) \leqslant Cn^{-1/3}\int_{0}^{\infty}\frac{y^{\lambda+1}}{(t^{2}+(\frac{2y}{\pi(2n-1)})^{2})(e^{y}+e^{-y})}\,dy. \tag{3.16}$$

Further by (3.7), we have for $t \in [0, 1)$ and $\lambda > 0$,

$$I_{n,1}(t) = 2(1+O(n^{-1/3}))\int_{0}^{n^{1/3}}\frac{y^{\lambda+1}}{(t^{2}+(\frac{2y}{\pi(2n-1)})^{2})(e^{y}+e^{-y})}\,dy$$
$$= 2(1+O(n^{-1/3}))\int_{0}^{\infty}\frac{y^{\lambda+1}}{(t^{2}+(\frac{2y}{\pi(2n-1)})^{2})(e^{y}+e^{-y})}\,dy. \tag{3.17}$$

Combining (3.13) with (3.9), (3.14), (3.16) and (3.17), we then obtain (2.4).

If $0 < |t| < 1$, $\lambda > -2$, then (3.15) implies

$$I_{n,2}(t) \leqslant Cn^{-1/3}C_1(\lambda+2)/t^{2}. \tag{3.18}$$

Next by (3.7),

$$I_{n,1}(t) = (2/t^{2})(1+O(n^{-1/3}))\left(\int_{0}^{n^{1/3}}\frac{y^{\lambda+1}}{e^{y}+e^{-y}}\,dy - \alpha_n(t)\right), \tag{3.19}$$

where

$$\alpha_n(t) := \int_{0}^{n^{1/3}}\frac{y^{\lambda+1}(\frac{2y}{\pi(2n-1)})^{2}}{(t^{2}+(\frac{2y}{\pi(2n-1)})^{2})(e^{y}+e^{-y})}\,dy \leqslant Cn^{-4/3}C_1(\lambda+2)/t^{2}. \tag{3.20}$$

It follows from (3.14), (3.18)–(3.20) that

$$I_n(t) = (2C_1(\lambda+2)/t^2)(1+\alpha_n^*(t)), \tag{3.21}$$

where $|\alpha_n^*(t)| \leqslant Cn^{-1/3}(1+(nt^2)^{-1})$.

Thus (3.9), (3.13) and (3.21) yield (2.6). This establishes statements (a) and (c).

(b) and (d). The proof is similar to that of statements (a) and (c). Let $t \in [-1,1]$, $\lambda > 0$, and $N = 2n, n \in \mathbb{N}$. Then the nodes $x_{n-k+1} = 1 - 2(k/n)^2$, $k = 0, 1, \ldots, n$, satisfy the conditions of Lemma 1(b) for $m = n+1 > s > 0$, and by (3.2),

$$(1-x)^s - P_n(x) = (1/\pi)\sin \pi s\,(1-x)\prod_{k=1}^{n}(x-1+2(k/n)^2)$$
$$\times \int_1^\infty \frac{(z-1)^{s-1}}{(z-x)\prod_{k=1}^{n}(z-1+2(k/n)^2)}\,dz.$$

Making the substitutions $z = 1+2(y/(\pi n))^2, x = 1-2t^2$, and $s = \lambda/2$, and taking account of the identity

$$L_{2n}(f_\lambda, t) = 2^{-s}P_n(1-2t^2),$$

we arrive at

$$|t|^\lambda - L_{2n}(f_\lambda, t) = (2/\pi)\sin(\pi\lambda/2)(\pi n)^{-\lambda}\prod_{k=1}^{n}(1-(nt/k)^2)$$
$$\times \int_0^\infty \frac{t^2 y^{\lambda-1}}{(t^2+(\frac{y}{\pi n})^2)\prod_{k=1}^{n}(1+\frac{y^2}{\pi^2k^2})}\,dy.$$

Next using (3.8), we can find the asymptotic behavior of the integral $I_n(t)$ on the right-hand side of this identity similarly to (3.14), (3.16) and (3.17) if $0 \leqslant |t| < 1$ and similarly to (3.18), (3.19) and (3.21) if $0 < |t| < 1$. Finally by (3.10), we obtain (2.5) and (2.7). □

4. Proofs of Theorems 1–3, 5 and 6

Proof of Theorem 1. Choosing $t = 0$ in (2.4), we obtain

$$L_{2n-1}(f_\lambda, 0) = (4/\pi)\sin(\pi\lambda/2)(\pi(2n-1)/2)^{-\lambda}C_1(\lambda)(1+O(n^{-1/3})).$$

Hence (2.1) follows. □

Proof of Theorem 2. Asymptotic formulae (2.6) and (2.7) show that strong asymptotics (2.2) and (2.3) immediately follow from the relations

$$\limsup_{n\to\infty}|\cos(\pi(2n-1)t/2)| = c(t), \qquad \limsup_{n\to\infty}|\sin(\pi nt)| = s(t), \tag{4.1}$$

where $0 < |t| < 1$.

To prove (4.1), we consider the following cases:

Case 1: We first suppose that t is irrational. Then the sequences $\{nt(\text{mod }1)\}_{n=1}^{\infty}$ and $\{(2n-1)t/2(\text{mod }1)\}_{n=1}^{\infty}$ are dense in $[0,1]$ by [6], there exist two increasing subsequences $\{n_k(t)\}_{k=1}^{\infty}$ and $\{r_k(t)\}_{k=1}^{\infty}$ of indices such that

$$\lim_{k\to\infty}|\cos(\pi(2n_k-1)t/2)| = \lim_{k\to\infty}|\sin(\pi r_k t)| = 1. \tag{4.2}$$

Case 2: Now, let

$$t = p/m, \quad (p,m) = 1, \quad |p|\in\mathbb{N}, \quad m\in\mathbb{N}, \quad 1\leqslant|p|<m. \tag{4.3}$$

If m is even, then for $r_k(t) := mk + m/2$,

$$|\sin(\pi r_k t)| = 1, \qquad k = 1,2,\dots\ . \tag{4.4}$$

Further, if m is odd, then by elementary properties of congruence modulo m [6], there exists $d_1(t)\in\mathbb{N}$, $1\leqslant d_1(t)\leqslant m-1$, satisfying $d_1p\equiv[m/2]+1\ (\text{mod }m)$. Hence we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}|\sin(\pi nt)| &= \sup_{0\leqslant d\leqslant m-1}|\sin(\pi dp/m)| = \sin(\pi([m/2]+1)/m)\\ &= \sin(\pi d_1 p/m)\\ &= \cos\frac{\pi}{2m}.\end{aligned} \tag{4.5}$$

Note too that for $r_k(t) := mk + d_1(t)$,

$$|\sin(\pi r_k t)| = \cos\frac{\pi}{2m}, \qquad k = 1,2,\dots\ .$$

Together with (4.2), (4.4) and (4.5) this yields the second relation in (4.1).

Assume now that (4.3) holds and p is even. Then for $n_k(t) := (1 + m(2k-1))/2$,

$$|\cos(\pi(2n_k-1)t/2)| = 1, \qquad k = 1,2,\dots\ . \tag{4.6}$$

Further, if p is odd, then there exists $d_2(t)\in\mathbb{N}$, $1\leqslant d_2(t)\leqslant m-1$, satisfying $(2d_2-1)p\equiv 2m-1\ (\text{mod }2m)$. Hence we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}|\cos(\pi(2n-1)t/2)| &= \sup_{1\leqslant d\leqslant m-1}\left|\cos\left(\pi\frac{(2d-1)p}{2m}\right)\right| = \left|\cos\left(\pi\frac{(2d_2-1)p}{2m}\right)\right|\\ &= \left|\cos\left(\pi\frac{2m-1}{2m}\right)\right| = \cos\frac{\pi}{2m}.\end{aligned} \tag{4.7}$$

Note too that for $n_k(t) := mk + d_2(t)$,

$$|\cos(\pi(2n_k-1)t/2)| = \cos\frac{\pi}{2m}, \qquad k = 1,2,\dots\ . \tag{4.8}$$

Then (4.2), (4.6), (4.7) and (4.8) yield the first relation in (4.1). This proves the theorem. □

Proof of Theorem 5. We first note that for $|t| = 1 - \beta_n/n$, where $\lim_{n\to\infty}\beta_n = 0$, the following asymptotics hold as $n\to\infty$:

$$\prod_{k=1}^{n}\left(1-\left(\frac{2n-1}{2k-1}t\right)^2\right)$$
$$= \cos(\pi(2n-1)t/2)\frac{\Gamma(n(1+t)+(1-t)/2)\Gamma(n(1-t)+(1+t)/2)}{(\Gamma(n+1/2))^2}$$
$$= (-1)^{n+1}\sin\left(\frac{\pi\beta_n(2n-1)}{2n}\right)\frac{\Gamma(2n-\beta_n+\beta_n/(2n))}{(\Gamma(n+1/2))^2}(1+o(1))$$
$$= \frac{(-1)^{n+1}\pi\beta_n\sqrt{2\pi}(2n-\beta_n+\beta_n/(2n))^{2n-\beta_n+\beta_n/(2n)}(n+1/2)e^{2n+1}}{\sqrt{2n-\beta_n+\beta_n/(2n)}\,e^{2n-\beta_n+\beta_n/(2n)}(2\pi)\,(n+1/2)^{2n+1}}(1+o(1))$$
$$= \frac{(-1)^{n+1}e\pi^{1/2}\beta_n 2^{2n-1}n^{2n}}{(n+1/2)^{2n}\,n^{1/2+\beta_n}}(1+o(1)) = (-1)^{n+1}\pi^{1/2}\beta_n n^{-(1/2+\beta_n)}2^{2n-1}(1+o(1)).$$

Together with (3.13) and (3.21) this yields for $N = 2n-1\to\infty$,

$$|t|^\lambda - L_{2n-1}(f_\lambda, t)$$
$$= -(2/\pi)\sin(\pi\lambda/2)(\pi(2n-1)/2)^{-(\lambda+2)}$$
$$\times\prod_{k=1}^{n}\left(1-\left(\frac{2n-1}{2k-1}t\right)^2\right)(2C_1(\lambda+2)/t^2)\,(1+\alpha_n^*(t))$$
$$= (-1)^{(N+1)/2}\sin(\pi\lambda/2)C_1(\lambda+2)(\pi N/2)^{-(\lambda+5/2)}\alpha_N N^{-\alpha_N/2}2^{N+1}(1+o(1)).$$

Thus (2.8) follows. Similarly, for $|t| = 1-\beta_n/n$ with $\lim_{n\to\infty}\beta_n = 0$, we obtain the following asymptotics as $n\to\infty$:

$$\prod_{k=1}^{n}\left(1-\left(\frac{n}{k}t\right)^2\right) = \frac{\sin(\pi nt)}{\pi nt}(n!)^{-2}\Gamma(n(1+t)+1)\Gamma(n(1-t)+1)$$
$$= (-1)^{n+1}\frac{\sin(\pi\beta_n)\,\Gamma(2n-\beta_n+1)}{\pi n\,(n!)^2}(1+o(1))$$
$$= \frac{(-1)^{n+1}(\beta_n/n)\sqrt{2\pi}(2n-\beta_n+1)^{2n-\beta_n+1}e^{2n}}{(2n-\beta_n+1)^{1/2}e^{2n-\beta_n+1}(2\pi)\,n^{2n+1}}(1+o(1))$$
$$= (-1)^{n+1}\pi^{-1/2}\beta_n n^{-(3/2+\beta_n)}\,2^{2n}(1+o(1)).$$

Now, (2.9) is a consequence of the corresponding asymptotics for $|t|^\lambda - L_{2n}(f_\lambda, t)$. □

Proof of Theorem 6. Let $N = 2n-1$ and let $t_N\in(-1,1)$ satisfy the equality

$$\Delta_{N,\lambda} = |\,|t_N|^\lambda - L_N(f_\lambda, t_N)|. \tag{4.9}$$

We first prove that

$$\lim_{N=2n-1\to\infty} N(1-t_N^2)=0. \tag{4.10}$$

Note that for any increasing sequence $\{N_k\}_{k=1}^{\infty}$ of positive odd numbers, the lower estimate

$$\Delta_{N_k,\lambda} \geqslant C N_k^{-(\lambda+5/2)} 2^{N_k}/\log N_k \tag{4.11}$$

holds for all large enough N_k. This follows from (2.8) if we set $\alpha_N = 1/\log N$.

Next, we have for large enough odd N that $|t_N| \in [1/2, 1]$. Indeed, if there exists a sequence $\{t_{N_k}\}_{k=1}^{\infty}$ satisfying $|t_{N_k}| \leqslant 1/2$, $k=1,2,\dots$, then it follows from (2.4) and the monotonicity of $\psi_N(t) := ((1+t)^{1+t}(1-t)^{1-t})^{N/2}$ on the interval $[0,1)$ that

$$\Delta_{N_k,\lambda} \leqslant C N_k^{-\lambda} \psi_{N_k}(1/2) \leqslant C N_k^{-\lambda} \left(\sqrt{3\sqrt{3}/2}\right)^{N_k} \leqslant C N_k^{-\lambda} (1.14)^{N_k}, \quad k=1,2,\dots .$$

This is a contradiction to (4.11).

Further, we need the following property of φ_N: for each $B>0$ there exists N_0 such that for $N>N_0$, φ_N is increasing in $[0, 1-B/N]$. Indeed,

$$(\varphi_N^2(t))' = ((1+t)^{1+t}(1-t)^{1-t})^N \left(-2t + N(1-t^2)\log\frac{1+t}{1-t}\right),$$

and for $0<t^2\leqslant 1-1/N$,

$$-2t + N(1-t^2)\log\frac{1+t}{1-t} > 2t(N(1-t^2)-1) \geqslant 0.$$

If $1-1/N < t^2 \leqslant (1-B/N)^2$, then for all large N

$$-2t + N(1-t^2)\log\frac{1+t}{1-t} \geqslant -2t + (2B - B^2/N)\log N > 0.$$

This yields the property.

To prove (4.10), we assume that there exists an increasing sequence $N_k = 2n_k - 1$, such that $|t_{N_k}| \geqslant 1/2$, $k=1,2,\dots$, and $\inf_{k\in\mathbb{N}} N_k(1-t_{N_k}^2) \geqslant A > 0$.

Then $1/2 \leqslant |t_{N_k}| \leqslant 1 - A/(2N_k)$, and it follows from (2.6) and the monotonicity of φ_{N_k} in $[0, 1-A/(2N_k)]$ that for all large odd N_k,

$$\Delta_{N_k,\lambda} \leqslant C(1+\alpha_{N_k,3}(t_{N_k})) N_k^{-(\lambda+2)} \varphi_{N_k}(t_{N_k}) \leqslant C N_k^{-(\lambda+2)} \varphi_{N_k}(1-A/(2N_k)). \tag{4.12}$$

Note that $\alpha_{N_k,3}(t_{N_k})$ in (4.12) is uniformly bounded because $N_k(1-t_{N_k}^2) \geqslant A > 0$. Now using the inequality

$$\varphi_{N_k}(1-y) \leqslant \sqrt{2}\, 2^{N_k} y^{(yN_k+1)/2}, \qquad y \in (0,1),$$

for $y = A/(2N_k)$, we obtain from (4.12)

$$\Delta_{N_k,\lambda} \leqslant C N_k^{-(\lambda+5/2+A/4)} 2^{N_k}.$$

Since $A>0$, this inequality is a contradiction to (4.11). Thus (4.10) follows.

Furthermore, setting $\gamma_N := N(1-t_N^2)$, we have $t_N = 1 - \alpha_N/N$, where $\alpha_N := N(1-\sqrt{1-\gamma_N/N})$. Then by (4.10), $\lim_{N=2n-1\to\infty} \alpha_N = \lim_{N=2n-1\to\infty} \gamma_N = 0$.

Applying now Theorem 5(a) for $t = t_N$, we obtain

$$\lim_{N=2n-1\to\infty} \frac{\Delta_{N,\lambda}}{\alpha_N N^{-(\lambda+5/2+\alpha_N/2)} 2^N} = 2|\sin(\pi\lambda/2)| C_1(\lambda+2)(\pi/2)^{-(\lambda+5/2)}. \tag{4.13}$$

Since for $N > 1$,

$$\alpha_N N^{-\alpha_N/2} \leqslant \max_{y\geqslant 0} yN^{-y/2} = (2/\log N)N^{-1/\log N} = (2/e)(\log N)^{-1},$$

we deduce from (4.13)

$$\limsup_{N=2n-1\to\infty} \frac{\Delta_{N,\lambda}}{N^{-(\lambda+5/2)} 2^N/\log N} \leqslant A_1, \tag{4.14}$$

where A_1 is defined by (2.11).

On the other hand, choosing $t_N^* = 1 - (2N\log N)^{-1}$ and using (2.8) for $\alpha_N = (2\log N)^{-1}$, we obtain

$$\liminf_{N=2n-1\to\infty} \frac{\Delta_{N,\lambda}}{N^{-(\lambda+5/2)} 2^N/\log N} \geqslant \lim_{N=2n-1\to\infty} \frac{\left| |t_N^*|^\lambda - L_N(f_\lambda, t_N^*)\right|}{N^{-(\lambda+5/2)} 2^N/\log N} = A_1. \tag{4.15}$$

Thus (4.14) and (4.15) yield (2.10) for $N = 2n-1$. Similarly, using (2.5), (2.7), (2.9), and (2.12), we arrive at (2.10) for $N = 2n$. □

Proof of Theorem 3. We first note that the theorem follows from statements (c) and (d) of Theorem 4 if we prove that the limit relations

$$\lim_{n\to\infty} |\cos(\pi(2n-1)t/2)|^{1/(2n-1)} = 1, \tag{4.16}$$

$$\lim_{n\to\infty} |\sin(\pi n t)|^{1/(2n)} = 1 \tag{4.17}$$

hold for a.e. $t \in [-1, 1]$.

To prove (4.16) and (4.17), we need the following fact from the metrical theory of diophantine approximation [10]: for a.e. $t \in [-1, 1]$ the set of all solutions (k, n) to the inequality $|t - k/n| \leqslant n^{-3}$ is finite.

This statement implies that for a.e. $t \in [-1, 1]$ there exists $n_0(t)$ such that for every rational number k/n with $n > n_0(t)$, the following inequality holds:

$$n|t - k/n| \geqslant n^{-2}. \tag{4.18}$$

Next, let $k = k(t)$ be a closest integer to nt. Then $n|t - k/n| \leqslant 1/2$, and using (4.18) for a.e. $t \in [-1, 1]$ and $n > n_0(t)$, we have

$$|\sin(\pi n t)| = \sin(\pi n |t - k/n|) \geqslant 2n|t - k/n| \geqslant 2n^{-2}.$$

Hence (4.17) follows.

Similarly, if $k_1 = k_1(t)$ is the closest odd integer to $(2n-1)t$, then $(2n-1)|t - k_1/(2n-1)| \leqslant 1$. Using (4.18) for a.e. $t \in [-1, 1]$ and $2n - 1 > n_0(t)$, we obtain

$$\left|\cos\left(\frac{\pi(2n-1)t}{2}\right)\right| = \sin\left(\frac{\pi(2n-1)}{2}\left|t - \frac{k_1}{2n-1}\right|\right) \geqslant (2n-1)\left|t - \frac{k_1}{2n-1}\right| \geqslant (2n-1)^{-2}.$$

This yields (4.16). □

Acknowledgments

The author thanks Michael Revers for inspiring discussions and the referee for helpful suggestions.

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